HEAT TRANSFER IN A LAYER IN THE PRESENCE OF MOVING HEAT SOURCES

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Inzhenerno-Fizicheskii Zhurnal, Vol. 14, No. 3, pp. 511–519, 1968

UDC 536.46

The problem regarding the distribution of temperatures is solved for the case of a layer in which a combustion front is moving at a constant speed, said front caused by the burnout of a fuel uniformly distributed through the layer.

We are dealing here with a problem that is characteristic of the heating of an agglomerated-charge layer in which a relatively uniformly distributed solid fuel burns out with the passage of time.

Before the onset of fuel combustion, the layer is first heated by a constant-temperature gas over a certain period of time adequate to achieve the stable ignition of the fuel and to heat the layer through a specified depth. Subsequently, from the instant of time taken as zero, air at a temperature T_0 is passed through the layer, and it is in this flow of air that the fuel is burned. The combustion front of the fuel moves through the layer at a constant speed w_c in the direction of the air flow.

The speed of motion for the combustion front and the law of heat generation in the combustion zone are functions of the concentration of the reacting gas and of its velocity, of the fuel concentration in the layer, of the temperature, etc., and they can be determined by solving a system of heat- and mass-transfer equations. However, this is an extremely complex problem. Here we will consider a simpler case and, namely, the case in which the velocity of the combustion front and the heat-generation law in the combustion zone are specified.

§1. Let us first consider the case in which the width of the heat-generation zone is equal to zero, i.e., the fuel burns out instantaneously. In this case, the initial temperature of the material is equal to zero. We will reckon the coordinate for the height of the layer from the point at which the air enters the layer (x = 0).

To determine the temperatures of the gas and the material in the layer, we have the following system of equations:

$$-fc_g\rho_g \frac{\partial T}{\partial \tau} - fc_g\rho_g v \frac{\partial T}{\partial x} = \alpha_v(T-t) - Q_i^c k\,\delta\left(\tau - \frac{x}{w_c}\right), \qquad (1')$$

$$\rho c(1-f)\frac{\partial t}{\partial \tau} = \alpha_v(T-t) \qquad (2')$$

with the boundary conditions

$$x=0,\quad T=T_0,$$
$$\tau = x/v,\quad t=0. \qquad (3')$$

It is assumed in the formulation of the problem that the longitudinal heat conduction is negligibly small. This condition is not always satisfied. If the material is finely dispersed, we can apparently assume that T = t, but the longitudinal heat conduction in the material cannot be neglected [1]. Moreover, we assume that the volume of the gases in passage through the combustion zone does not change, and that the porosity is constant in time and through the height of the layer. In first approximation these conditions are acceptable, since it is assumed that the quantity of fuel in the layer is small in volume (for example, for an agglomerated charge the quantity of fuel in the layer does not exceed 10% by volume).

Let us introduce new variables and, namely, we will reckon the time at each point through the height of the layer from the instant at which this point is reached by the gas entering the layer at the instant $\tau = 0$:

$$\tau_1 = \tau - \frac{x}{v}.$$

The system then assumes the form

$$-\frac{\partial T}{\partial x}\frac{c_g\rho_g f v}{\alpha_v} = T - t - \frac{Q_i^c k}{\alpha_v}\,\delta\left[\tau_1 - x\left(\frac{1}{w_c}-\frac{1}{v}\right)\right], \qquad (1)$$

$$\frac{\partial t}{\partial \tau_1}\frac{\rho c(1-f)}{\alpha_v} = T - t \qquad (2)$$

with the boundary conditions

$$\tau_1 = 0,\quad t=0,$$
$$x=0,\quad T=T_0. \qquad (3)$$

We will apply the Laplace transform with respect to τ_1; we obtain the images

$$-\frac{d\bar T}{dx}\frac{vc_g\rho_g f}{\alpha_v} = \bar T - \bar t - \frac{Q_i^c k}{\alpha_v}\exp\left[-x\left(\frac{1}{w_c}-\frac{1}{v}\right)s\right],$$

$$s\bar t\,\frac{\rho c(1-f)}{\alpha_v} = \bar T - \bar t. \qquad (4)$$

Let us turn to the new variables

$$y = \frac{\alpha_v x}{vc_g\rho_g f},\quad z = \frac{\alpha_v\tau_1}{\rho c(1-f)}.$$

The solution of system (4) for $\bar T$ will be

$$\bar T = \frac{T_0}{s}\exp\left(-y + \frac{y}{1+s}\right) + \qquad (5)$$

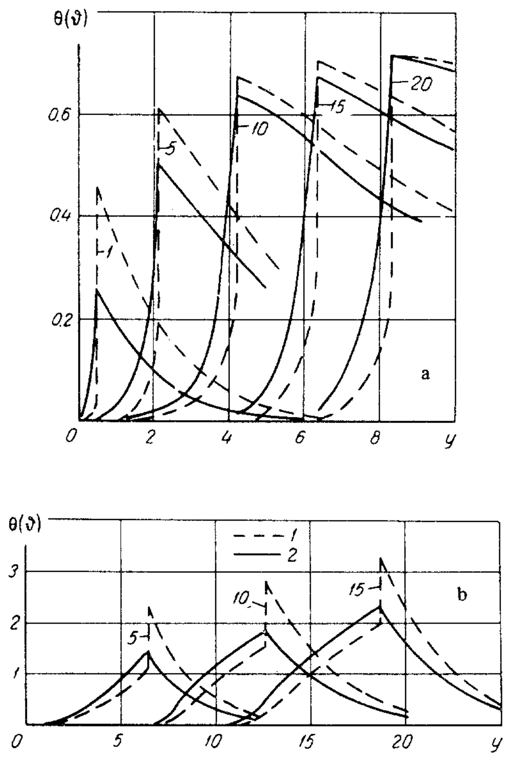

Fig. 1. Temperature distribution of gas (1) and material (2) along bed height at various time instants at N > 1 (a) and N < 1 (b) (figures on curves show value of z).

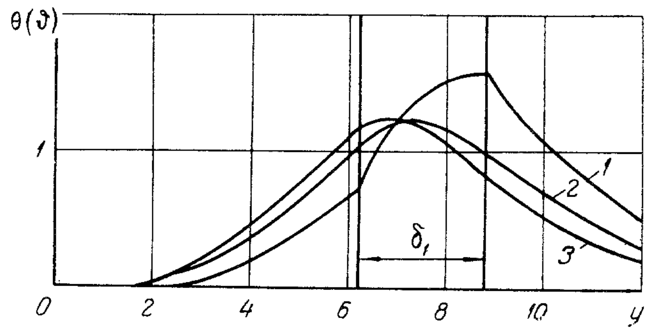

Fig. 2. Temperature distribution of gas (1) and material (2, 3) along bed height (N < 1): 1, 2) regular heat generation ($\varphi(\beta) = 1/\delta_1$); 3) heat generation according to the law $\varphi(\beta) = \frac{2}{\delta_1^2}\beta$.

$$+\frac{Q_i^c k}{\rho c(1-f)}\frac{1+s}{(Ns-1)(1+s)+1}\times$$

$$\times\left[\exp\left(-y+\frac{y}{1+s}\right)-\exp(-yNs)\right],$$

where

$$N=\frac{vc_g\rho_g f}{\rho c(1-f)}\left(\frac{1}{w_c}-\frac{1}{v}\right)\approx\frac{vc_g\rho_g f}{\rho c(1-f)w_c}=\frac{W_r}{W_m},$$

since $v \gg w_c$.

Hence we can see that the solution is the sum of two solutions, one of which is the solution for the problem relating to the heating of a nonmoving layer by a gas with an initial zero temperature T_0; the second solution is the one formulated for the problem involving a gas temperature equal to zero at the inlet.

Since we know the solution of the first problem [2], we will seek the preimage only for the second term $(T_2; t_2)$.

After simple transformations, for this we will obtain

$$T_2=\frac{Q_i^c k}{\rho c(1-f)}\left\{\int_0^z\left[\frac{\exp(-\tau-y)}{N-1}+\right.\right. \tag{6}$$

$$\left.+\frac{\exp\left[-y-\frac{\tau}{N}-z\left(1-\frac{1}{N}\right)\right]}{N^2(1-N)}\right]\times$$

$$\times I_0(2\sqrt{y\tau})\,d\tau-$$

$$-\sigma_0(z-yN)\left[\frac{N-\exp\left[-(z-yN)\left(1-\frac{1}{N}\right)\right]}{N(N-1)}\right]+$$

$$\left.+\exp(-z-y)\frac{I_0(2\sqrt{yz})}{N}\right\}.$$

Here $\sigma_0(z-yN)$ is the unit impulsive function [3]

$$\sigma_0(z-yN)=\begin{cases}0 & \text{for } z<yN,\\ 1 & \text{for } z>yN.\end{cases} \tag{7}$$

We see from (6) that the gas temperature exhibits a discontinuity at the point $z = yN$. Experiencing a rise in temperature in passage through the charge layer and cooling off after passage of the combustion front through the layer, the gas exhibits a jumpwise rise in temperature at the front by an amount $Q_i^c k/\rho c(1-f)N$, and the temperature of the gas then falls, since the heat is expended on the heating of the charge approaching the combustion front.

The temperature of the gas at the combustion front (after the burning up of the fuel) will be

$$T_2=\frac{Q_i^c k}{\rho c(1-f)}\times \tag{8}$$

$$\times\left\{\int_0^z\left[\frac{\exp\left(-\tau-\frac{z}{N}\right)}{N-1}-\frac{\exp\left(-\frac{z}{N}-z\right)}{N^2(N-1)}\right]\times\right.$$

$$\times I_0\left(2\sqrt{\frac{z\tau}{N}}\right)d\tau-$$

$$\left.-\frac{1}{N}\exp\left(-z-\frac{z}{N}\right)I_0\left(2\sqrt{\frac{z^2}{N}}\right)\right\}.$$

It can be demonstrated that as z approaches infinity the temperature of the gas at the combustion front is given by

$$\lim_{\substack{z\to\infty\\ N\gg 1}} T=\frac{Q_i^c}{\rho c(1-f)}\,\frac{1}{N-1}, \tag{9}$$

$$\lim_{\substack{z\to\infty\\ N\ll 1}} T=\frac{Q_i^c}{\rho c(1-f)}\,\frac{1}{N(1-N)}. \tag{10}$$

It is easy to see that these temperature values correspond to the conditions of a steady-state counterflow.

Let us now determine the temperature of the material. For this we will use Eq. (1), but in dimensionless form

$$-\frac{\partial T}{\partial y}=T-t-\frac{Q_i^c k}{\rho c(1-f)}\,\delta(z-yN). \tag{1''}$$

Here we employed the property of the delta function [4]

$$\delta(ax)=\frac{\delta(x)}{|a|}.$$

In the differentiation of Eq. (8) we used the formula

$$\frac{d}{dy}\sigma_0(z-yN)=-N\delta(z-yN).$$

Finally, for the charge temperature we have

$$t_2=\frac{Q_i^c k}{\rho c(1-f)}\times$$

$$\times\left\{\int_0^z\left[\frac{\exp(-y-\tau)}{N-1}+\frac{\exp\left[-y-z-\frac{1}{N}(\tau-z)\right]}{N^2(1-N)}\right]\times\right.$$

$$\times I_1(2\sqrt{y\tau})\sqrt{\frac{\tau}{y}}\,d\tau+$$

$$+\exp(-y-z)\frac{I_1(2\sqrt{yz})}{N}\sqrt{\frac{z}{y}}-$$

$$\left.-\sigma_0(z-yN)\frac{1-\exp\left[-(z-yN)\left(1-\frac{1}{N}\right)\right]}{N-1}\right\}. \tag{11}$$

We see from this expression that the charge-temperature function is continuous, since the multiplier for $\sigma_0(z-yN)$ at the point $z = yN$ vanishes.

The results from the numerical calculations of T_2 and t_2 for $N > 1$ and $N < 1$ are shown in Fig. 1.

§2. In actual processes the liberation of heat does not occur instantaneously, but in proportion to the burning out of the fuel particles. Consequently, the combustion (heat generation) zone is finite in width (Δ).

The solution of this problem may be regarded as a superposition of solutions with sources in the form of delta functions which depend on the variable β, with respect to which the summation (integration) is carried out.

Thus we first have to find the solution of the following problem (we write the equations in dimensionless form):

$$-\frac{\partial T}{\partial y}=T-t-\frac{Q_i^c k}{\rho c(1-f)}\,\delta(z-yN-\beta N_1), \quad (12)$$

$$\frac{\partial t}{\partial z}=T-t, \quad (2'')$$

$$z=0,\quad t=0,\quad y=0,\quad T=0. \quad (3'')$$

In the following we will assume that $N_1 \approx N$, since the velocity of the gases is considerably greater than the speed at which the combustion front is moving.

Let us note that the δ-function written in dimensional form

$$\delta\left(\tau-\frac{x+x_0}{w_c}\right)$$

indicates that the heat generation at the point $x = 0$ does not begin at the instant $\tau = 0$, but somewhat later and, namely, at the instant $\tau = x_0/w_c$.

On application of Laplace transforms, Eqs. (12) and (2) assume the form

$$-\frac{d\bar{T}}{dy}=\bar{T}-\bar{t}-\frac{Q_i^c k}{\rho c(1-f)}\exp(-yNs-\beta Ns),$$

$$s\bar{t}=\bar{T}-\bar{t}. \quad (13)$$

Having solved (13) for $\bar{T}$, we finally obtain

$$\bar{T}=\frac{Q_n^p k}{\rho c(1-f)}\left\{\exp(-\beta Ns-yNs)-\right.$$
$$\left.-\exp\left[-\beta Ns-y\left(1-\frac{1}{1+s}\right)\right]\right\}\times$$
$$\times\left\{1-\frac{1}{1+s}-Ns\right\}^{-1}. \quad (14)$$

This expression differs from (5) only in the multiplier $\exp(-\beta Ns)$, and, consequently, the original (according to the displacement theorem) will be

$$T(\beta)=\frac{Q_i^c k}{\rho c(1-f)}\left\{\sigma_0(z-\beta N)\int\limits_0^{z-\beta N}\left[\frac{\exp(-y-\tau)}{N-1}+\right.\right.$$
$$\left.+\frac{\exp\left[-y-\frac{1}{N}\tau-(z-\beta N)\left(1-\frac{1}{N}\right)\right]}{N^2(1-N)}\right]\times$$
$$\times I_0(2\sqrt{y\tau})\,d\tau-$$
$$-\sigma_0(z-\beta N-yN)\times$$
$$\times\left[\frac{N-\exp\left[-(z-yN-\beta N)\left(1-\frac{1}{N}\right)\right]}{N(N-1)}\right]+$$
$$+\sigma_0(z-\beta N)\exp\left[-(z-\beta N)-y\right]\times$$
$$\left.\times\frac{I_0(2\sqrt{y(z-\beta N)})}{N}\right\}. \quad (15)$$

If the generation of heat begins at $\tau = 0$, i.e., $\beta = 0$, we obtain (6).

However, if the heat generation occurs uniformly in a zone of thickness δ_1 ($\delta_1 = \alpha_v\Delta/vc_g\rho_g f$), the gas temperature is defined as the following integral:

$$T=\int\limits_0^{\delta_1}T(\beta)\,\frac{d\beta}{\delta_1}. \quad (16)$$

In calculating the gas temperature from (16) and (15), we use the following formulas:

$$\int\limits_0^{\delta}f(z-\beta N)\,\sigma_0(z-\beta N)\,d\beta=$$
$$=\frac{1}{N}\int\limits_0^{\delta N}\sigma_0(z-\beta N)\,f(z-\beta N)\,d\beta N=$$
$$=\frac{1}{N}\,\sigma_0(z)\int\limits_0^{z}f(\varepsilon)\,d\varepsilon-$$
$$-\frac{1}{N}\,\sigma_0(z-\delta N)\int\limits_0^{z-\delta N}f(\varepsilon)\,d\varepsilon \quad (17)$$

and

$$\int\limits_0^{\delta}\sigma_0(z-\beta N-yN)\,f(z-\beta N-yN)\,d\beta=$$
$$=\frac{1}{N}\,\sigma_0(z-yN)\int\limits_0^{z-yN}f(\varepsilon)\,d\varepsilon-$$
$$-\frac{1}{N}\,\sigma_0(z-yN-\delta N)\int\limits_0^{z-yN-\delta N}f(\varepsilon)\,d\varepsilon. \quad (17')$$

Substituting (15) into (16) and using (17) and (17'), we finally obtain

$$T=\frac{Q_i^c k}{\rho c(1-f)N\delta_1}\times$$
$$\times\left\{\sigma_0(z-\delta_1 N)\int\limits_0^{z-\delta_1 N}\left[(z-\delta_1 N-\tau)\,\frac{\exp(-\tau-y)}{1-N}-\right.\right.$$
$$-\frac{\exp\left[-y-\frac{1}{N}\tau-(z-\delta_1 N)\left(1-\frac{1}{N}\right)\right]}{N(N-1)^2}-$$
$$\left.-\frac{N-2}{(N-1)^2}\exp(-y-\tau)\right]I_0(2\sqrt{y\tau})\,d\tau-$$

$$-\sigma_0(z)\int_0^z\left[(z-\tau)\,\frac{\exp(-\tau-y)}{1-N}-\frac{\exp\left[-y-\frac{1}{N}\,\tau-z\left(1-\frac{1}{N}\right)\right]}{N(N-1)^2}-\frac{N-2}{(N-1)^2}\times\exp(-y-\tau)\right]I_0(2\sqrt{y\tau})\,d\tau+\sigma_0(z-yN)\times\left[\frac{z-yN}{1-N}-\frac{\exp\left[-(z-yN)\left(1-\frac{1}{N}\right)\right]}{(N-1)^2}+\frac{1}{(N-1)^2}\right]-\sigma_0(z-\delta_1N-yN)\left[\frac{z-yN-\delta_1N}{1-N}-\frac{\exp\left[-(z-yN-\delta_1N)\left(1-\frac{1}{N}\right)\right]-1}{(N-1)^2}\right]\Bigg\}. \quad (18)$$

For the case of a source in the form of a delta function in Eq. (12), i.e., for system (12)–(2″), we obtain the temperature of the material in the following form:

$$t(\beta)=\frac{Q_i^ck}{\rho c(1-f)}\Bigg\{\sigma_0(z-\beta N)\int_0^{z-\beta N}\left[\frac{\exp(-y-\tau)}{N-1}+\frac{\exp\left[-y-\frac{1}{N}\,\tau-(z-\beta N)\left(1-\frac{1}{N}\right)\right]}{N^2(1-N)}\right]\times I_0(2\sqrt{y\tau})\sqrt{\frac{\tau}{y}}\,d\tau+\sigma_0(z-\beta N)\exp[-y-(z-\beta N)]\times\frac{I_1(2\sqrt{y(z-\beta N)})}{N}\sqrt{\frac{z-\beta N}{y}}-\sigma_0(z-\beta N-yN)\times\frac{1-\exp\left[-(z-yN-\beta N)\left(1-\frac{1}{N}\right)\right]}{N-1}\Bigg\}. \quad (19)$$

For the case of heat generation in a zone of width δ_1 in analogy with (16) we obtain

$$t=\int_0^{\delta_1}t(\beta)\,\frac{d\beta}{\delta_1}. \quad (20)$$

After the calculations we will have

$$t=\frac{Q_i^ck}{\rho c(1-f)\,\delta_1N}\times\Bigg\{\sigma_0(z-\delta_1N)\int_0^{z-\delta_1N}\left[(z-\delta_1N-\tau)\,\frac{\exp(-\tau-y)}{1-N}-\frac{\exp\left[-y-\frac{\tau}{N}-(z-\delta_1N)\left(1-\frac{1}{N}\right)\right]}{(N-1)^2N}-\frac{N-2}{(N-1)^2}\exp(-y-\tau)\right]I_1(2\sqrt{y\tau})\times\sqrt{\frac{\tau}{y}}\,d\tau-\sigma_0(z)\int_0^z\left[(z-\tau)\,\frac{\exp(-\tau-y)}{1-N}-\frac{\exp\left[-y-\frac{\tau}{N}-z\left(1-\frac{1}{N}\right)\right]}{(N-1)^2N}-\frac{N-2}{(N-1)^2}\exp(-y-\tau)\right]\times I_1(2\sqrt{y\tau})\sqrt{\frac{\tau}{y}}\,d\tau+\sigma_0(z-yN)\times\left[\frac{z-yN}{1-N}-\frac{N}{(N-1)^2}\times\left(\exp\left[-(z-yN)\left(1-\frac{1}{N}\right)\right]-1\right)\right]-\sigma_0(z-yN-\delta_1N)\times\left[\frac{z-yN-\delta_1N}{1-N}-\frac{N}{(N-1)^2}\left(\exp\left[-(z-yN-\delta_1N)\times\left(1-\frac{1}{N}\right)\right]-1\right)\right]\Bigg\}. \quad (21)$$

Figure 2 shows the results from the calculation of the gas and material temperatures for a heat-generation zone of finite width for a single instant of time ($N < 1$).

In principle, it is possible to calculate the temperature field in the layer for any heat-generation law $\varphi(\beta)$ over the width of the zone, but for this we must have

$$\int_0^{\delta_1}\varphi(\beta)\,d\beta=1. \quad (22)$$

From (22), given uniform heat generation in the zone, we have

$$\varphi(\beta)=\text{const}=\frac{1}{\delta_1}.$$

In the general case, if we represent $\varphi(\beta)$ as proportional to β^n, from (22) we have

$$\varphi(\beta)=\frac{n+1}{\delta_1^{n+1}}\,\beta^n,$$

and the gas temperature will be

$$T = \frac{n+1}{\delta_1^{n+1}} \int_0^{\delta_1} T(\beta)\, \beta^n \, d\beta.$$

The temperature of the material is determined in analogous fashion.

Figure 2 shows the temperature curve for the material through the height of the bed in the case of heat generation proportional to the distance from the leading edge of the zone. We see from the figure that the temperature maximum is more pronounced in this case; in front of the heat-generation zone the temperature is higher than the temperature in the case of uniform heat generation; beyond that zone, however, it is lower.

§3. In the general case, when the layer was heated prior to the onset of the process of fuel burnout so that when $z = 0$, $t = f(y)$, and the air enters the layer at a temperature T_0, the solution will be the sum of the familiar Nusselt solution [5] and the one derived in §2.

NOTATION

T and t are the temperatures of the gas and material; $\theta = T\rho c(1-f)/Q_i^c k$ and $\vartheta = t\rho c(1-f)/Q_i^c k$ are the dimensionless temperatures of the gas and material; x is the bed height; τ is the time; $c\rho$ is the heat capacity of the material; $c_g\rho_g$ is the heat capacity of the gas; f is the porosity of the bed; v is the gas velocity in the bed; w_c is the velocity of the burning front; W_c and W_m are the water equivalents of the gas and material; α_v is the volumetric coefficient of heat transfer between the gas and material; Q_i^c is the calorific value of the fuel in the bed (with account for its burnout ratio and heat flow rate for structural transformations); k is the fuel concentration; $\delta(\tau - x/w_c$ is the delta-function; $y = \alpha_v x/vc_g\rho_g f$ $(\beta = \alpha_v x_0/vc_g\rho_g f)$ is the dimensionless height; $z = \alpha_v \tau_1/\rho c(1-f)$ is the dimensionless time; $I_\nu(2(y\tau)^{1/2})$ is the modified Bessel function of the first kind, of order ν; Δ is the width of heat generation zone; $\delta_1 = \alpha_v\Delta/vc_g\rho_g f$ is the dimensionless width of the heat generation zone; $\varphi(\beta)$ is the law of heat generation over the zone width.

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13 February 1967

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